# ON THE COMPLETE LAX TYPE INTEGRABILITY OF A GENERALIZED RIEMANN TYPE HYDRODYNAMIC SYSTEM

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ABSTRACT. The complete integrability of a generalized Riemann type hydrodynamic system is studied by means of a novel combination of symplectic and differential-algebraic tools. A compatible pair of polynomial Poissonian structures, a Lax representation and a related infinite hierarchy of conservation laws are constructed.

## 1. INTRODUCTION

We shall study the complete integrability of the dispersionless Riemann type hydrodynamic flow

$$(1.1) \qquad D_t^{N-1} u = \bar{z}_x^2, \qquad D_t \bar{z} = 0$$

on a $2\pi$-periodic functional manifold $\bar{M}^N \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^N)$, where $N \in \mathbb{N}$ is an arbitrary natural number, the vector $(u, D_t u, D_t^2 u, ..., D_t^{N-1} u, \bar{z})^\intercal \in \bar{M}^N$, the differentiations $D_x := \partial/\partial x$, $D_t := \partial/\partial t + u\partial/\partial x$ satisfy the Lie-algebraic commutator relationship

$$(1.2) \qquad [D_x, D_t] = u_x D_x.$$

and $t \in \mathbb{R}$ is an evolution parameter. The system can be considered as a slight generalization of the dispersionless Riemann hydrodynamic system (suggested recently by M. Pavlov and D. Holm [9]) in the form

$$(1.3) \qquad D_t^{N-1} u = \bar{z}, \qquad D_t \bar{z} = 0$$

for $N \in \mathbb{N}$ and extensively studied in $[1, 3, 4, 5, 2, 6]$, where it was proved that it is a Lax integrable bi-Hamiltonian flow on the manifold $\bar{M}^N$ and possesses an infinite hierarchy of mutually commuting dispersive Lax integrable Hamiltonian flows.

For the case $N = 2$ it is well known [8, 11] that the system (1.1) is a smooth Lax integrable bi-Hamiltonian flow on the $2\pi$-periodic functional manifold $\bar{M}^2$, whose Lax representation is given by the compatible linear system

$$(1.4) \qquad D_x f = \begin{pmatrix} \bar{z}_x & 0 \\ -\lambda(u + u_x/\bar{z}_x) & -\bar{z}_{xx}/\bar{z}_x \end{pmatrix} f, \qquad D_t f = \begin{pmatrix} 0 & 0 \\ -\lambda\bar{z}_x & u_x \end{pmatrix} f,$$

where $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^2)$ and $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter.

Our focus here is an investigation of the Lax integrability of the Riemann type hydrodynamic system (1.1) for $N = 3$ on a $2\pi$-periodic functional manifold $\bar{M}^3 \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^3)$ for a vector $(u, v, \bar{z})^\intercal \in \bar{M}^3$. We treat this problem in the following extended form:

$$(1.5) \qquad D_t u = v, \qquad D_t v = \bar{z}_x^2, \qquad D_t \bar{z} = 0.$$

The flow (1.5) can be recast as a one on a $2\pi$-periodic functional manifold $M^3 \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^3)$ for a vector $(u, v, z)^\intercal \in M^3$ as

$$(1.6) \qquad D_t u = v, \qquad D_t v = z, \qquad D_t z = -2z u_x,$$

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where, for further convenience, we have made the change of variables: $z := \bar{z}_x^2$. We will also use the form of the flow (1.6):

$$
(1.7) \qquad \begin{pmatrix} du/dt \\ dv/dt \\ dz/dt \end{pmatrix} = K[u,v,z] := \begin{pmatrix} v - uu_x \\ z - uv_x \\ -2u_x z - uz_x \end{pmatrix}
$$

defining a standard smooth dynamical system on the infinite-dimensional functional manifold $M^3$, where $K : M^3 \to T(M^3)$ is the corresponding smooth vector field on $M^3$.

In the sequel, we shall prove the following result using symplectic gradient-holonomic and differential algebraic tools.

**Proposition 1.1.** *The Riemann type hydrodynamic flow (1.7) is a bi-Hamiltonian dynamical system on the functional manifold $M^3$ with respect to two compatible Poissonian structures $\vartheta, \eta :$ $T^*(M^3) \to T(M^3)$*

$$
(1.8) \qquad \vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2} D_x z^{1/2} \end{pmatrix}, \eta := \begin{pmatrix} \partial^{-1} & u_x \partial^{-1} & 0 \\ \partial^{-1} u_x & v_x \partial^{-1} + \partial^{-1} v_x & \partial^{-1} z_x - 2z \\ 0 & z_x \partial^{-1} + 2z & 0 \end{pmatrix},
$$

*possessing an infinite hierarchy of mutually commuting conservation laws and a non-autonomous Lax representation of the form*

$$
(1.9) \qquad D_t f = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda z_x & u_x \end{pmatrix} f,
$$

$$
D_x f = \begin{pmatrix} \lambda^2 u \sqrt{z} & \lambda v \sqrt{z} & z \\ -\lambda^3 t u \sqrt{z} & -\lambda^2 t v \sqrt{z} & -\lambda t z \\ \lambda^4 (tuv - u^2) - & -\lambda v_x/\sqrt{z} + & \lambda^2 \sqrt{z}(u - tv) - \\ -\lambda^2 u_x/\sqrt{z} & +\lambda^3(tv^2 - uv) & -z_x/2z \end{pmatrix} f,
$$

*where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^3)$.*

## 2. SYMPLECTIC GRADIENT-HOLONOMIC INTEGRABILITY ANALYSIS

Our first steps in proving Proposition 1.1 are fashioned using the symplectic gradient-holonomic method, which takes us a long way towards the desired result.

2.1. **Poissonian structure analysis on the functional manifold $M^3$.** By employing the symplectic gradient-holonomic approach [8, 11, 12] to studying the integrability of smooth nonlinear dynamical systems on functional manifolds, one can find a set of conservation laws for (1.7) by constructing some solutions $\varphi := \varphi[u,v,z] \in T^*(M^3)$ to the functional Lax gradient equation:

$$
(2.1) \qquad d\varphi/dt + K'^{,*}\varphi = \mathrm{grad}\mathcal{L},
$$

where $\varphi' = \varphi'^{,*}$, $\mathcal{L} \in D(M^3)$ is a suitable Lagrangian functional and the linear operator $K'^{,*} :$ $T^*(M^3) \to T^*(M^3)$ is the adjoint with respect to the standard convolution $(\cdot, \cdot)$ on $T^*(M^3) \times T(M^3)$, of the Fréchet-derivative of a nonlinear mapping $K : M^3 \to T(M^3)$; namely,

$$
(2.2) \qquad K'^{,*} = \begin{pmatrix} uD_x & -v_x & z_x + 2zD_x \\ 1 & u_x + uD_x & 0 \\ 0 & 1 & -u_x + uD_x \end{pmatrix}.
$$

The Lax gradient equation (2.1) can be, owing to (1.3), rewritten as

$$
(2.3) \qquad D_t\varphi + k[u,v,z]\varphi = \mathrm{grad}\,\mathcal{L},
$$

where the matrix operator

$$
(2.4) \qquad k[u,v,z] := \begin{pmatrix} 0 & -v_x & z_x + 2zD_x \\ 1 & u_x & 0 \\ 0 & 1 & -u_x \end{pmatrix}.
$$

The first vector elements

$$(2.5) \qquad \varphi_\vartheta[u,v,z] = (z - uv_x, -v + uu_x, u), \mathcal{L}_\vartheta = 0$$
$$\varphi_\eta[u,v,z] = (v_x, -u_x, -1)^\mathsf{T}, \mathcal{L}_\eta = 0,$$
$$\varphi_0[u,v,z] = (-(u_x z^{-1/2})_x, (z^{-1/2})_x, (v_x/2 - u_x^2/4)z^{-3/2})^\mathsf{T}, \mathcal{L}_0 = 0,$$

as can be easily checked, are solutions of the functional equation (2.3). From an application of the standard Volterra homotopy formula

$$(2.6) \qquad H := \int_0^1 d\mu(\varphi[\mu u, \mu v, \mu z], (u,v,z)^\mathsf{T}),$$

one finds the conservation laws for (1.3); namely,

$$(2.7) \qquad H_\eta = \frac{1}{2}\int_0^{2\pi} dx(2uz - v^2 - u^2 v_x),$$

$$H_\vartheta := \int_0^{2\pi} dx(uv_x/2 - vu_x/2 - z), \quad H_0 := \frac{1}{2}\int_0^{2\pi} dx(u_x^2 - 2v_x)z^{-1/2}.$$

It is now quite easy, making use of the conservation laws (2.7), to construct a Poissonian structure $\vartheta : T^*(M^3) \to T(M^3)$ for the dynamical system (1.7). If we use the representations

$$(2.8) \qquad H_\vartheta = \int_0^{2\pi} dx(uv_x/2 - vu_x/2 - z) := (\psi_\vartheta, (u_x, v_x, z_x)^\mathsf{T}),$$

$$\psi_\vartheta := (-v/2, u/2, z^{-1/2}D_x^{-1}z^{1/2}/2)^\mathsf{T},$$

it follows that the vector $\psi_\vartheta \in T^*(M^3)$ satisfies the Lax gradient equation (2.3):

$$(2.9) \qquad D_t\psi_\vartheta + k[u,v,z]\psi_\vartheta = \operatorname{grad} \mathcal{L}_\vartheta,$$

where the Lagrangian function $\mathcal{L}_\vartheta = (\psi_\vartheta, K) - H_\vartheta$. Thus, based on the inverse co-symplectic functional expression

$$(2.10) \qquad \vartheta^{-1} := \psi_\vartheta' - \psi_\vartheta'^* = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z^{-1/2}D_x^{-1}z^{-1/2}/2 \end{pmatrix}$$

one readily obtains the linear co-symplectic operator on the manifold $M^3$:

$$(2.11) \qquad \vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2}D_x z^{1/2} \end{pmatrix},$$

which is the corresponding Poissonian operator for the dynamical system (1.2). It is also important to observe that the dynamical system (1.2) is a Hamiltonian flow on the functional manifold $M^3$ with respect to the Poissonian structure (2.11).

$$(2.12) \qquad K[u,v,z] = -\vartheta \operatorname{grad} H_\eta.$$

2.2. **Poissonian structure analysis on $\bar{M}^3$.** In what follows, we shall find it convenient to construct we will construct other Poissonian structures for dynamical system (1.5) on the manifold $\bar{M}^3$, rewritten in the equivalent form

$$(2.13) \qquad \frac{d}{dt}\begin{pmatrix} u \\ v \\ \bar{z} \end{pmatrix} = \bar{K}[u,v,\bar{z}] := \begin{pmatrix} v - uu_x \\ \bar{z}_x^2 - uv_x \\ 0 \end{pmatrix},$$

where $\bar{K} : \bar{M}^3 \to T(\bar{M}^3)$ is the corresponding vector field on $\bar{M}^3$. To proceed, we need to obtain additional solutions to the Lax gradient equation (2.3) on the functional manifold $\bar{M}^3$

$$(2.14) \qquad D_t\bar{\psi} + \bar{k}[u,v,z]\bar{\psi} = \operatorname{grad} \bar{\mathcal{L}},$$

where the matrix operator is

$$(2.15) \qquad \bar{k}[u, v, \bar{z}] := \begin{pmatrix} 0 & -v_x & -\bar{z}_x \\ 1 & u_x & 0 \\ 0 & -2\partial \, \bar{z}_x & u_x \end{pmatrix},$$

and which we may rewrite in the componentwise form

$$(2.16) \qquad D_t \bar{\psi}^{(1)} = v_x \bar{\psi}^{(2)} + \bar{z}_x \bar{\psi}^{(3)} + \delta \bar{\mathcal{L}}/\delta u,$$
$$D_t \bar{\psi}^{(2)} = -\bar{\psi}^{(1)} - u_x \bar{\psi}^{(2)} + \delta \bar{\mathcal{L}}/\delta v,$$
$$D_t \bar{\psi}^{(3)} = 2(\bar{z}_x \bar{\psi}^{(2)})_x - u_x \bar{\psi}^{(3)} + \delta \bar{\mathcal{L}}/\delta \bar{z},$$

where the vector $\bar{\psi} := (\bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)})^\intercal \in T^*(\bar{M}^3)$. As a simple consequence of (2.16), one obtains the following system of differential relationships:

$$(2.17) \qquad \begin{aligned} D_t^3 \tilde{\psi}^{(2)} &= -2\bar{z}_x^2 \tilde{\psi}_x^{(2)} + D_t^2 \partial^{-1}(\delta \bar{\mathcal{L}}/\delta v) - \\ &\quad -\partial^{-1} < \mathrm{grad} \, \bar{\mathcal{L}}, (u_x, v_x, \bar{z}_x)^\intercal >, \\ D_t \tilde{\psi}^{(2)} &= -\tilde{\psi}^{(1)} + \partial^{-1}(\delta \bar{\mathcal{L}}/\delta v), \\ D_t \tilde{\psi}^{(3)} &= 2\bar{z}_x \tilde{\psi}_x^{(2)} + \partial^{-1}(\delta \bar{\mathcal{L}}/\delta \bar{z}). \end{aligned}$$

Here we have defined $(\bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)})^\intercal := (\tilde{\psi}_x^{(1)}, \tilde{\psi}_x^{(2)}, \tilde{\psi}_x^{(3)})^\intercal$ and made use of the commutator relationship for differentiations $D_t$ and $D_x$ :

$$(2.18) \qquad [D_t, \alpha^{-1} D_x] = 0,$$

which holds for the function $\alpha := 1/\bar{z}_x$, where $D_t \bar{z} = 0$. It therefore follows that after solving the first equation of system (2.17), one can recursively sole the remaining two equations. In particular, it is easy to see that the three vector elements

$$(2.19) \qquad \begin{aligned} \tilde{\psi}_0 &= (-v, u, -2\bar{z}_x)^\intercal, \quad \bar{\mathcal{L}}_0 = 0; \\ \tilde{\psi}_\theta &= (-u_x/\bar{z}_x, 1/\bar{z}_x, (u_x^2 - 2v_x)/(2\bar{z}_x^2))^\intercal, \bar{\mathcal{L}}_\theta = 0; \\ \tilde{\psi}_\eta &= (u/2, -x/2, \partial^{-1}[(2v_x - u_x^2)/(2\bar{z}_x)]), \bar{\mathcal{L}}_\eta = (D_x \tilde{\psi}_\eta, \bar{K}) - H_\vartheta, \end{aligned}$$

are solutions of the system (2.17). The first two elements of (2.19) lead to the Volterra symmetric vectors $\bar{\psi}_0 = D_x \tilde{\psi}_0, \bar{\psi}_\theta = D_x \tilde{\psi}_\theta \in T^*(\bar{M}^3) : \bar{\psi}_0' = \bar{\psi}_0'^{,*}, \bar{\psi}_\theta' = \bar{\psi}_\theta'^{,*}$ entailing the trivial conservation laws $(\bar{\psi}_0, \bar{K}) = 0 = (\bar{\psi}_\theta, \bar{K})$. The third element of (2.19) gives rise to the Volterra asymmetric vector $\bar{\psi}_\eta := D_x \tilde{\psi}_\eta : \bar{\psi}_\eta' \neq \bar{\psi}_\eta'^{,*}$, entailing the following inverse co-symplectic functional expression:

$$(2.20) \qquad \bar{\eta}^{-1} := \bar{\psi}_\eta' - \bar{\psi}_\eta'^{,*} = \begin{pmatrix} \partial & 0 & -\partial \frac{u_x}{\bar{z}_x} \\ 0 & 0 & \partial \frac{1}{\bar{z}_x} \\ -\frac{u_x}{\bar{z}_x}\partial & \frac{1}{\bar{z}_x}\partial & \frac{u_x}{2\bar{z}_x}\partial \frac{u_x}{\bar{z}_x} - \\ & & -\frac{v_x}{\bar{z}_x}\partial \frac{1}{\bar{z}_x} - \frac{1}{\bar{z}_x}\partial \frac{v_x}{\bar{z}_x} \end{pmatrix}.$$

Correspondingly, the Poissonian operator $\bar{\eta} : T^*(\bar{M}^3) \to T(\bar{M}^3)$ is

$$(2.21) \qquad \bar{\eta} = \begin{pmatrix} \partial^{-1} & u_x \partial^{-1} & 0 \\ \partial^{-1} u_x & v_x \partial^{-1} + \partial^{-1} v_x & \partial^{-1} \bar{z}_x \\ 0 & \bar{z}_x \partial^{-1} & 0 \end{pmatrix},$$

subject to which the following Hamiltonian representation

$$(2.22) \qquad \bar{K}[u, v, \bar{z}] = -\bar{\eta} \, \mathrm{grad} \, H_\eta |_{z=z_x^2}$$

holds on the manifold $\bar{M}^3$.

2.3. **Hamiltonian integrability analysis.** Next, we return to our integrability analysis of the dynamical system (1.7) on the functional manifold $M^3$. It is easy to recalculate the form of the Poissonian operator (2.21) on the manifold $\bar{M}^3$ to that acting on the manifold $M^3$, giving rise to the second Hamiltonian representation of (1.7):

$$(2.23) \qquad K[u, v, z] = -\eta \, \mathrm{grad} \, H_\vartheta,$$

where $\eta : T^*(M^3) \to T(M^3)$ is the corresponding Poissonian operator. As a next important point, the Poissonian operators (2.11) and (2.21) are compatible [10, 8, 11, 7] on the manifold $\bar{M}^3$; that

is, the operator pencil $(\vartheta + \lambda \eta) : T^*(M^3) \to T(M^3)$ is also Poissonian for arbitrary $\lambda \in \mathbb{R}$. As a consequence, any operator of the form

$$(2.24) \qquad \vartheta_n := \vartheta(\vartheta^{-1}\eta)^n$$

for all $n \in \mathbb{Z}$ is Poissonian on the manifold $M^3$. Using now the homotopy formula (2.6) and recursion property of the Poissonian pair (2.12) and (2.21), it is easy to construct the related infinite hierarchy of mutually commuting conservation laws

$$(2.25) \qquad \begin{aligned} \gamma_j &= \int_0^1 d\mu (\text{grad } \gamma_j[\mu u, \mu v, \mu z], (u, v, z)^\mathsf{T}), \\ \text{grad } \gamma_j[u, v, z] &:= \Lambda^j \text{grad } H_\eta, \end{aligned}$$

for the dynamical system (1.7), where $j \in \mathbb{Z}_+$ and $\Lambda := \vartheta^{-1}\eta : T^*(M^3) \to T^*(M^3)$ is the corresponding recursion operator, which satisfies the so called associated Lax commutator relationship

$$(2.26) \qquad d\Lambda/dt = [\Lambda, K'^{,*}].$$

In the course of above analysis and observations, we have proved the following result.

**Proposition 2.1.** *The Riemann hydrodynamic system (1.7) is a bi-Hamiltonian dynamical system on the functional manifold $M^3$ with respect to the compatible Poissonian structures $\vartheta, \eta : T^*(M^3) \to T(M^3)$*

$$(2.27) \qquad \vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2}D_x z^{1/2} \end{pmatrix}, \eta := \begin{pmatrix} \partial^{-1} & u_x \partial^{-1} & 0 \\ \partial^{-1}u_x & v_x \partial^{-1} + \partial^{-1}v_x & \partial^{-1}z_x - 2z \\ 0 & z_x \partial^{-1} + 2z & 0 \end{pmatrix}$$

*and possesses an infinite hierarchy of mutually commuting conservation laws (2.25).*

Concerning the existence of an additional infinite and parametrically $\mathbb{R} \ni \lambda$-ordered hierarchy of conservation laws for the dynamical system (1.2), it is instructive to consider the dispersive nonlinear dynamical system

$$(2.28) \qquad \begin{pmatrix} du/d\tau \\ dv/d\tau \\ dz/d\tau \end{pmatrix} = -\vartheta \text{ grad } H_0[u, v, z] := \begin{pmatrix} -(z^{-1/2})_x \\ -(u_x z^{-1/2})_x \\ z^{1/2}(\frac{u_x^2 - 2v_x}{2z})_x \end{pmatrix} = \tilde{K}[u, v, z].$$

By solving the corresponding Lax equation

$$(2.29) \qquad d\tilde{\varphi}/dt + \tilde{K}'^{,*}\tilde{\varphi} = 0$$

for an element $\tilde{\varphi} \in T^*(M^3)$ in a suitably chosen asymptotic form, one can construct an infinite ordered hierarchy of conservation laws for (1.2), which we will not delve into here. This hierarchy and the existence of an infinite and parametrically $\mathbb{R} \ni \lambda$-ordered hierarchy of conservation laws for the Riemann type dynamical system (1.2) provided compelling indications that it is completely integrable in the sense of Lax on the functional manifold $M^3$. We will be the complete integrability in the next section using rather powerful differential-algebraic tools that were devised recently in [1, 2, 6].

## 3. Differential-algebraic integrability analysis: $N = 3$

Consider a polynomial differential ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x, t\}\}$ generated by a fixed functional variable $u \in \mathbb{R}\{\{x, t\}\}$ and invariant with respect to two differentiations $D_x := \partial/\partial x$ and $D_t := \partial/\partial t + u\partial/\partial x$ that satisfy the Lie-algebraic commutator relationship (1.2)

$$(3.1) \qquad [D_x, D_t] = u_x D_x$$

together with the constraint (1.6) expressed in the differential-algebraic functional form

$$D_t^3 u = -2D_t^2 u D_x u.$$

Since the Lax representation for the dynamical system (1.7) can be interpreted [1, 11] as the existence of a finite-dimensional invariant ideal $\mathcal{I}\{u\} \subset \mathcal{K}\{u\}$ realizing the corresponding finite-dimensional representation of the the Lie-algebraic commutator relationship (3.1), this ideal can be constructed as

$$(3.2) \qquad \mathcal{I}\{u\} := \{\lambda^2 u f_1 + \lambda v f_2 + z^{1/2}f_3 \in \mathcal{K}\{u\} : f_j \in \mathcal{K}, 1 \le j \le 3, \lambda \in \mathbb{R}\},$$

where $v = D_t u$, $z = D_t^2 u$ and $\lambda \in \mathbb{R}$ is an arbitrary real parameter. To find finite-dimensional representations of the $D_x$- and $D_t$-differentiations, it is necessary [1] first to find the $D_t$-invariant kernel $\ker D_t \subset \mathcal{I}\{u\}$ and next to check its invariance with respect to the $D_x$-differentiation. It is easy to show that

$$(3.3) \qquad \ker D_t = \{f \in \mathcal{K}^3\{u\} : D_t f = q(\lambda)f, \quad \lambda \in \mathbb{R}\},$$

where the matrix $q(\lambda) := q[u, v, z; \lambda] \in \mathrm{End}\,\mathcal{K}\{u\}^3$ is given as

$$(3.4) \qquad q(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda z_x & u_x \end{pmatrix}.$$

To obtain the corresponding representation of the $D_x$-differentiation in the space $\mathcal{K}^3$, it suffices to find a matrix $l(\lambda) := l[u, v, z; \lambda] \in \mathrm{End}\,\mathcal{K}\{u\}^3$ that

$$(3.5) \qquad D_x f = l(\lambda)f$$

for $f \in \mathcal{K}\{u\}^3$ and the related ideal

$$(3.6) \qquad \mathcal{R}\{u\} := \{< g, f >_{\mathcal{K}^3}: f \in \ker D_t \subset \mathcal{K}^3\{u\}, \ g \in \mathcal{K}^3\}$$

is $D_x$-invariant with respect to the matrfix differentiation representation 3.5). Straightforward calculations using this invariance condition then yield the following matrix

$$(3.7) \qquad l(\lambda) = \begin{pmatrix} \lambda^2 u\sqrt{z} & \lambda v\sqrt{z} & z \\ -\lambda^3 tu\sqrt{z} & -\lambda^2 tv\sqrt{z} & -\lambda tz \\ \lambda^4(tuv - u^2) - & -\lambda v_x/\sqrt{z} + & \lambda^2\sqrt{z}(u - tv) - \\ -\lambda^2 u_x/\sqrt{z} & +\lambda^3(tv^2 - uv) & -z_x/2z \end{pmatrix}$$

entering the linear equation 3.5). Thus, the following proposition is proved.

**Proposition 3.1.** *The generalized Riemann type dynamical system (1.6) is a bi-Hamiltonian integrable flow possessing a non-autonomous Lax representation of the form*

$$(3.8) \qquad D_t f = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda z_x & u_x \end{pmatrix} f,$$

$$D_x f = \begin{pmatrix} \lambda^2 u\sqrt{z} & \lambda v\sqrt{z} & z \\ -\lambda^3 tu\sqrt{z} & -\lambda^2 tv\sqrt{z} & -\lambda tz \\ \lambda^4(tuv - u^2) - & -\lambda v_x/\sqrt{z} + & \lambda^2\sqrt{z}(u - tv) - \\ -\lambda^2 u_x/\sqrt{z} & +\lambda^3(tv^2 - uv) & -z_x/2z \end{pmatrix} f,$$

*where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^3)$.*

*Remark* 3.2. Simple analogs of the above differential-algebraic calculations for the case $N = 2$ lead readily to the corresponding Riemann type hydrodynamic system

$$(3.9) \qquad D_t u = \bar{z}_x^2, \qquad D_t \bar{z} = 0$$

on the functional manifold $\bar{M}^2$, which possesses the following matrix Lax representation:

$$(3.10) \qquad D_t f = \begin{pmatrix} 0 & 0 \\ -\lambda \bar{z}_x & u_x \end{pmatrix}, \qquad D_x f = \begin{pmatrix} \bar{z}_x & 0 \\ -\lambda(u + u_x/\bar{z}_x) & -\bar{z}_{xx}/\bar{z}_x \end{pmatrix} f,$$

where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^2)$.

As one can readily see, these differential-algebraic results provide a direct proof of Proposition 1.1 describing the integrability of system (1.7) for $N = 3$. The matrices (3.7) are not of standard form since they depend explicitly on the temporal evolution parameter $t \in \mathbb{R}$. Nonetheless, the matrices (3.4) and (3.7) satisfy for all $\lambda \in \mathbb{R}$ the well-known Zakharov-Shabat type compatibility condition

$$(3.11) \qquad D_t l(\lambda) = [q(\lambda), l(\lambda)] + D_x l(\lambda) - u_x l(\lambda),$$

which follows from the Lax type relationships (3.3) and (3.5)

$$(3.12) \qquad D_t f = q(\lambda)f, \qquad D_x f = l(\lambda)f$$

and the commutator condition (3.1). Moreover, taking into account that the dynamical system (1.7) has a compatible Poissonian pair (2.11) and (2.21) depending only on the variables $(u, v, z)^\mathsf{T} \in M^3$

and not depending on the temporal variable $t \in \mathbb{R}$, one can certainly assume that it also possesses a standard autonomous Lax representation, which can possibly be found by means of a suitable gauge transformation of (3.12). We plan to pursue this line of analysis in a forthcoming paper.

## 4. Concluding remarks

A new nonlinear Hamiltonian dynamical system representing Riemann type hydrodynamic equation (1.1) in two and three dimensions proves to be a very interesting example of a Lax integrable dynamical system, as we have proved here. In particular, the integrability prerequisites of this dynamical system, such as compatible Poissonian structures, an infinite hierarchy of conservation laws and related Lax representation have been constructed by means of both the symplectic gradient-holonomic approach [8, 11, 12] and innovative differential-algebraic tools devised recently [1, 4] for analyzing the integrability of a special infinite hierarchy of Riemann type hydrodynamic systems. It is also quite clear from recent research in this area and our work in this paper that the dynamical system (1.1) is a Lax integrable bi-Hamiltonian flow for arbitrary integers $N \in \mathbb{N}$; this is perhaps most readily verified by means of the differential-algebraic approach, which was devised and successfully applied here for the cases $N = 2$ and 3.

We have seen in the course of this investigation that perhaps the most important lesson that one can derive from this approach is the following: If an investigation of a given nonlinear Hamiltonian dynamical system via the gradient-holonomic method indicates (but does not necessarily prove) that the system is Lax integrable, then its Lax representation, can often be shown to exist and then successfully derived by means of a suitably constructed invariant differential ideal $\mathcal{I}\{u\}$ of the ring $\mathcal{K}\{u\}$ in accordance with the differential-algebraic approach developed here for the integrability analysis of the Riemann hydrodynamical system investigated above. Consequently, when it comes applying this lesson to the investigation of other nonlinear dynamical systems, it is natural to start with systems that are known to be Lax integrable and to try to identify and characterize those algebraic structures responsible for the existence of a related finite-dimensional matrix representation for the basic $D_x$- and $D_t$-differentiations in a vector space $\mathcal{K}^p$ for some finite $p \in \mathbb{Z}_+$.

It seems plausible that if one could do this for several classes of Lax integrable dynamical systems, certain patterns in the algebraic structures may be detected that can be used to assemble a more extensive array of symplectic and differential-algebraic tools capable of resolving the question of complete integrability for many other types of nonlinear Hamiltonian dynamical systems. Moreover, if the integrability is established in this manner, the approach should also serve as a means of constructing associated artifacts of the integrability such as Lax representations and hierarchies of mutually commuting invariants. As a particular differential-algebraic problem of interest concerning these matrix representations, one can seek to develop a scheme for the effective construction of functional generators of the corresponding invariant finite-dimensional ideals $\mathcal{I}\{u\} \subset \mathcal{K}\{u\}$ under given differential-algebraic constraints imposed on the $D_x$- and $D_t$-differentiations.

We have demonstrated here that an approach combining the gradient-holonomic method with some recently devised differential-algebraic techniques can be a very effective and efficient way of investigating integrability for a particular class of infinite-dimensional Hamiltonian dynamical systems (generalized Riemann hydrodynamical systems). But a closer look at the specific details of the approach employed here reveals, we believe, that this combination of methods can be adapted to perform effective integrability analysis of a much wider range of dynamical systems - a goal that we intend to pursue in the very near future.

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